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Diagrams, fission spaces and global Lie theory

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I'll recall how to construct algebraic Poisson varieties by gluing pieces of surface with wild boundary conditions (extending the q-Hamiltonian framework), and then move on to discuss the link to quiver varieties and how this may be generalised, leading to a theory of "diagrams" for the wild character varieties (i.e. the wild nonabelian Hodge moduli spaces in their Betti algebraic structure). Much of this is motivated by quite straightforward questions about classifying rational Lax representations of finite dimensional integrable systems.

Some references:
--The first examples of fission spaces were really in Birkhoff's 1913 paper; they were shown to be q-Hamiltonian in arXiv:math/0203161, but they weren't given this name until Ann. Inst. Fourier 59, 7 (2009), and the story was then extended to the general case in arXiv:1111.6228, arXiv:1512.08091 (joint with Yamakawa)
--The theory of diagrams is in arXiv:1907.11149 (with Yamakawa), and has been extended by Doucot arXiv:2107.02516.

Diagrams, fission spaces & global Lie theory

Philip Boalch, IMJ-PRG & CNRS Paris

- new results joint with Daisuke Yamakawa (Tokyo Univ. Science) arxiv: 1907.11149, CRAS 2020
- or by Jean Douçot arxiv: 2107.02516
- see also short survey arxiv: 1703 for more background

Lie theory

$$\mathfrak{g} \longrightarrow G$$

$$X \longmapsto \exp(X)$$

Connection $\quad \dfrac{X}{2\pi i}\dfrac{dz}{z} \longmapsto$ monodromy

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moduli spaces: $\quad \mathcal{M}^* \longrightarrow \mathcal{M}_B \quad$ wild character variety (same dimension)

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$$\begin{array}{ccccc} \mathcal{M}^* & \hookrightarrow & \mathcal{M}_{DR} & \xrightarrow{\cong} & \mathcal{M}_B \\ & & & \searrow^{\cong} & \\ & & & & \mathfrak{M} \end{array}$$

$\mathcal{M}_B$: wild character variety (same dimension)

$\mathfrak{M}$: wild harmonic bundles (2d self-duality)

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moduli spaces:

$$\begin{tikzcd} \mathcal{M}^* \arrow[r, hook] & \mathcal{M}_{DR} \arrow[r, "\cong"] \arrow[d, leftrightarrow, "\cong", "\text{wild nonabelian Hodge}"'] \arrow[dr, "\cong"] & \mathcal{M}_B \\ & \mathcal{M}_{Dol} \arrow[r, "\cong"] & \mathfrak{M} \end{tikzcd}$$

$\mathcal{M}_B$: wild character variety (same dimension)

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wild nonabelian Hodge

meromorphic Higgs bundles

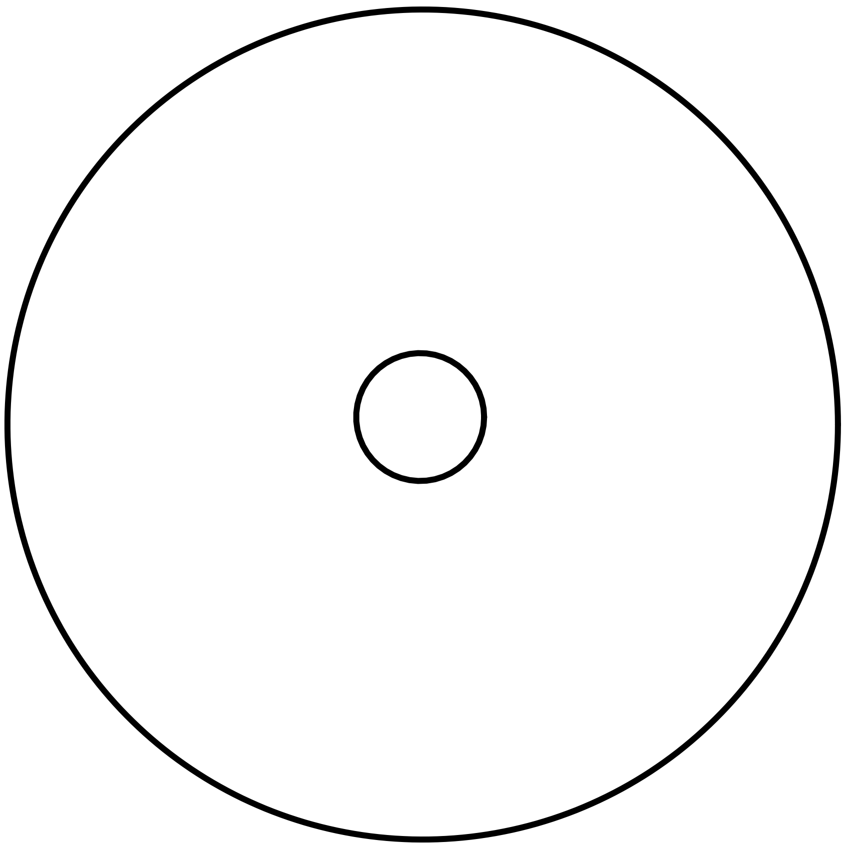
wild character variety (same dimension)

wild harmonic bundles (2d self-duality)

"Nonabelian Hodge space"

Classify via diagrams? (e.g. sometimes $\mathcal{M}^*$ is a quiver variety)

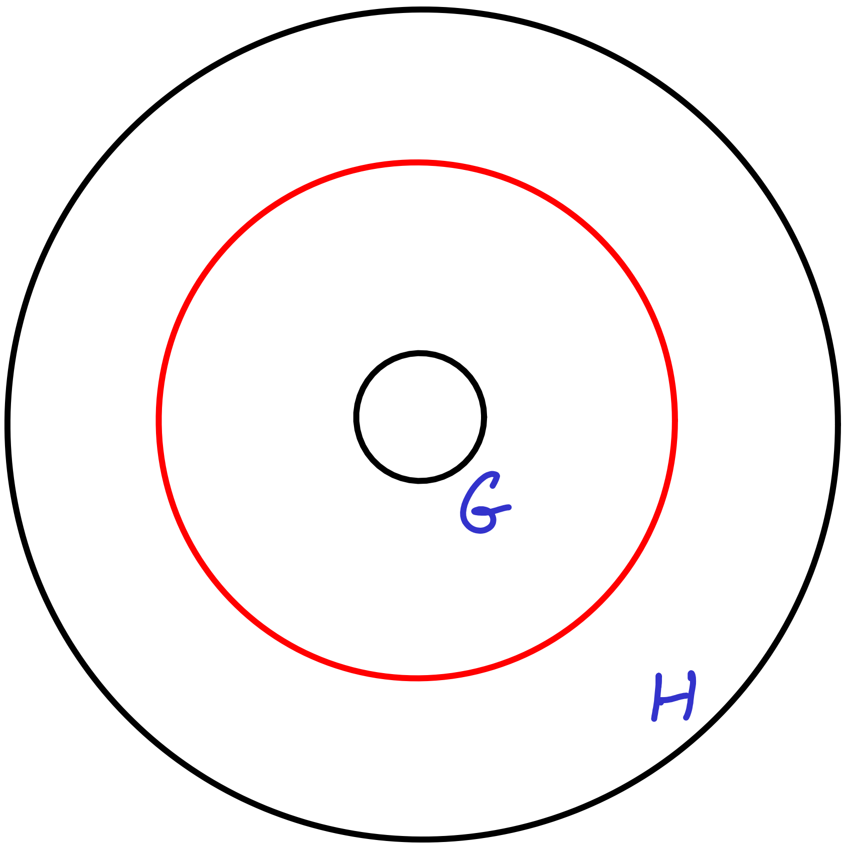
Fission spaces



Fission spaces

$V = \bigoplus_{i \in I} V_i$ $\quad$ I graded vector space

$G = GL(V) \supset H = GrAut(V) \cong \prod GL(V_i)$



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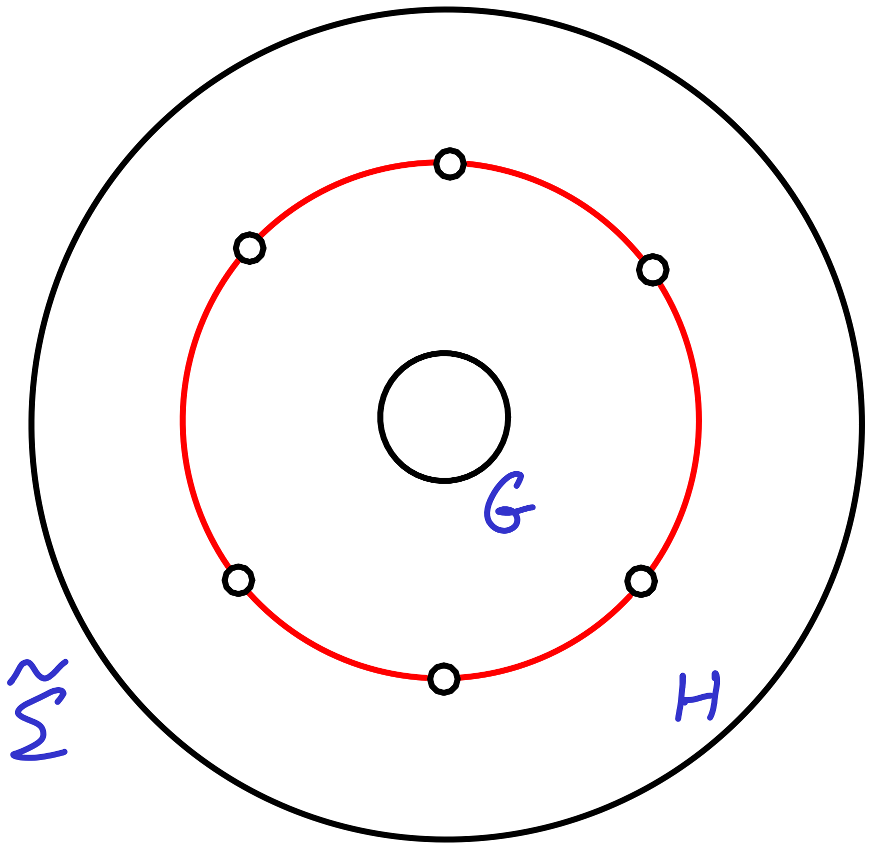

- $2k$ tangential punctures $\circ$

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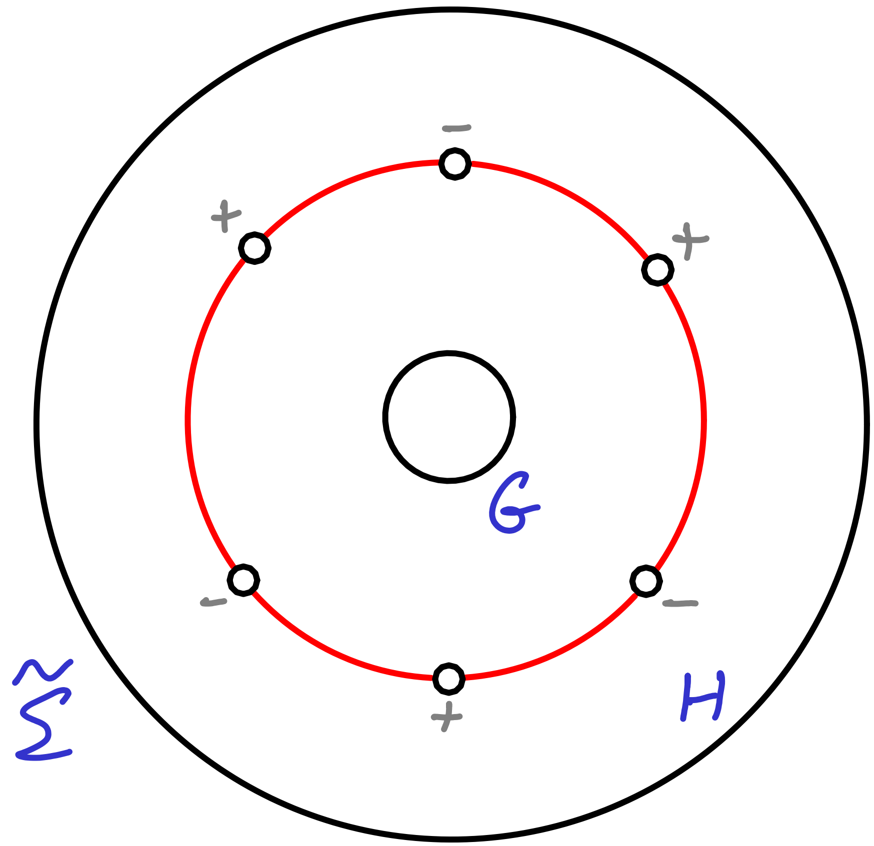

- $2k$ tangential punctures $\circ$
- $U_+, U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \subset G$ (Stokes groups)

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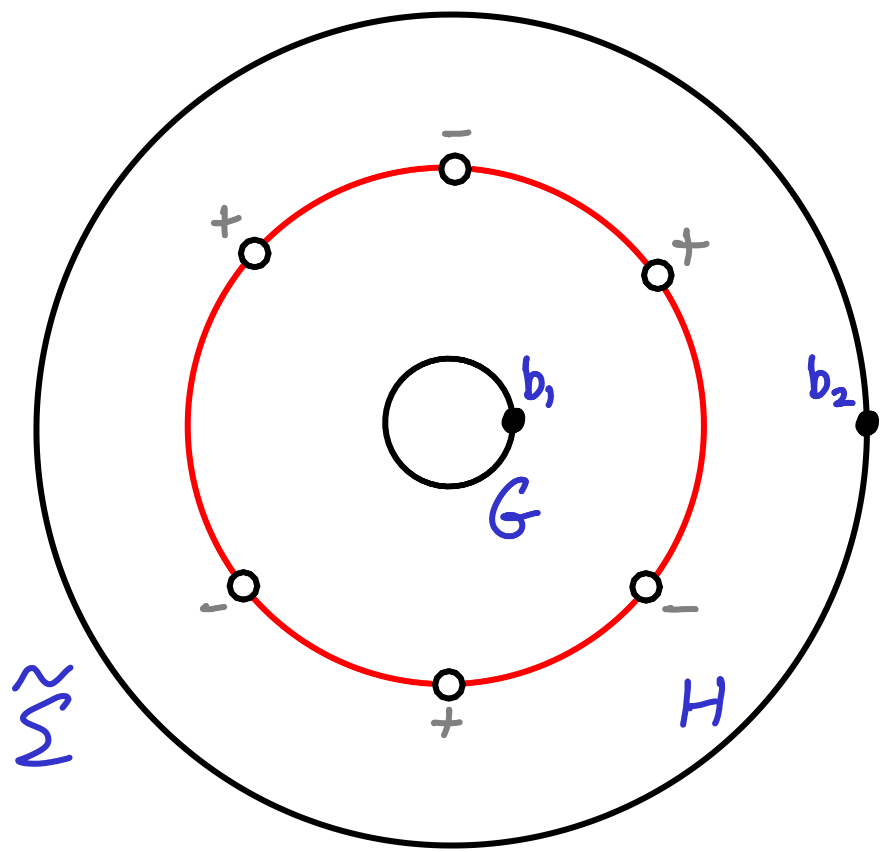

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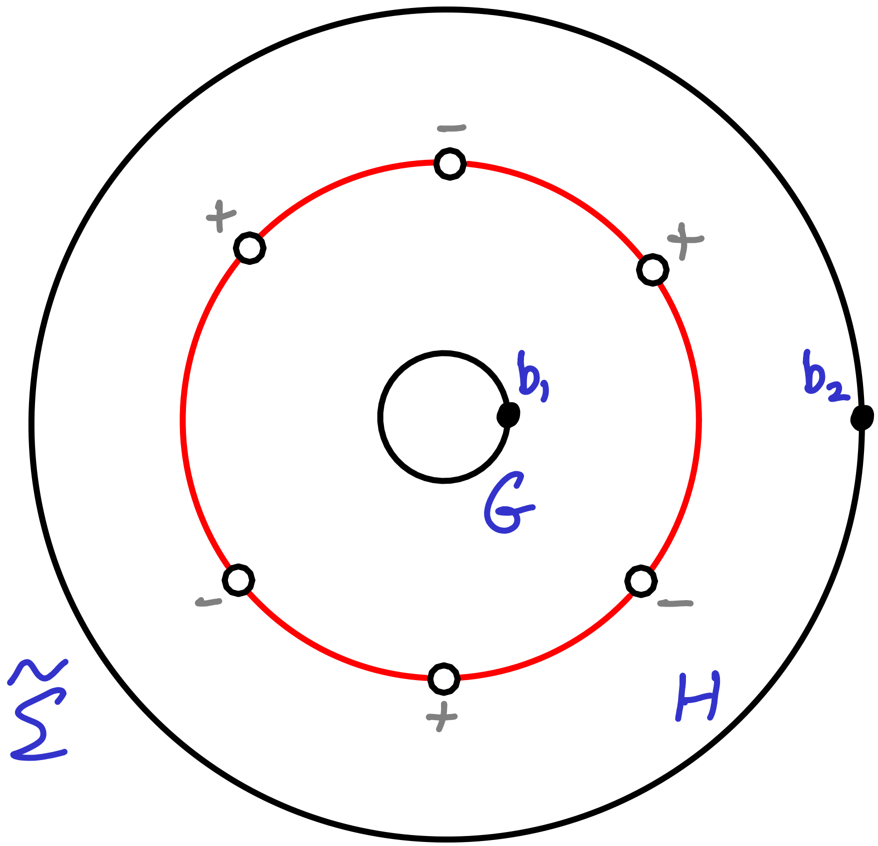

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- $\mathcal{A} = {}_G\mathcal{A}^k_H = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

$\cong G \times H \times (U_+ \times U_-)^k$

$\cong \{\text{Stokes local systems framed at } b_1, b_2\} / \text{iso.}$

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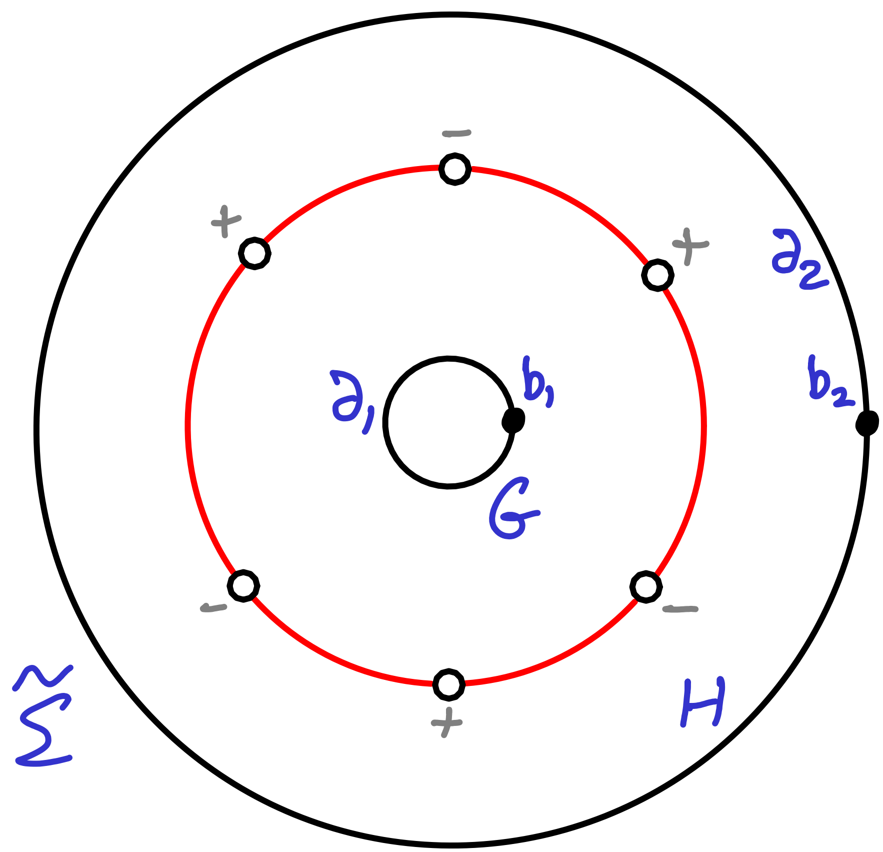

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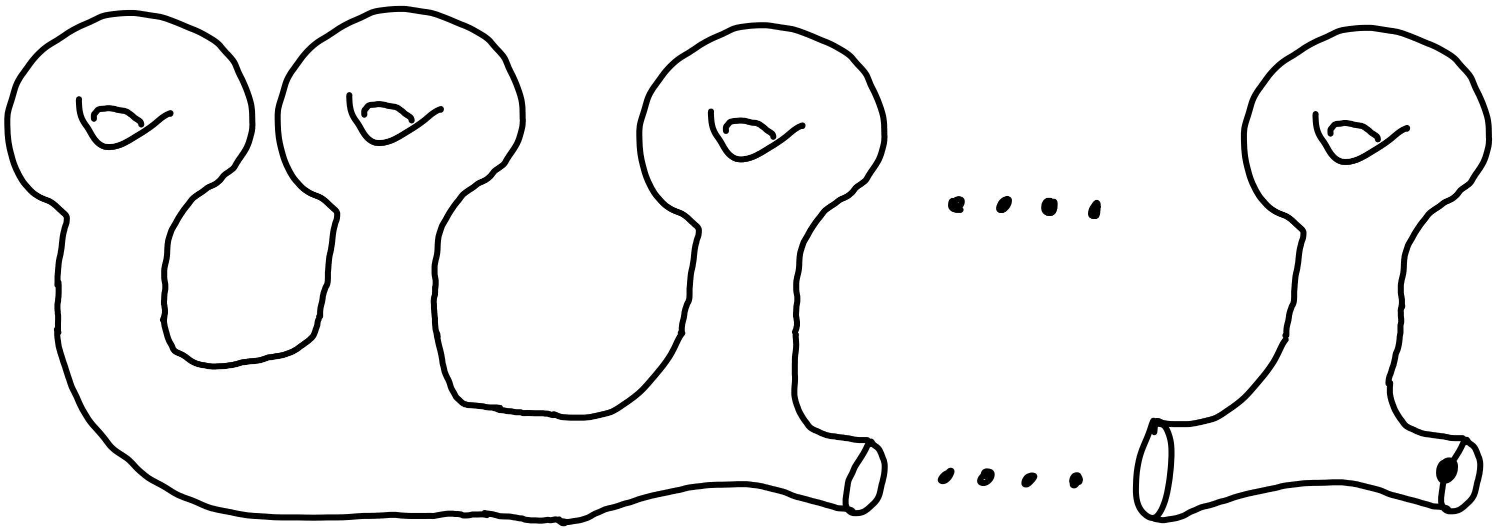
Thm $\mathcal{A}$ is a quasi-Hamiltonian $G \times H$ space with moment map $\mu: \mathcal{A} \to G \times H$, $\quad \mu(\rho) = (\rho(\partial_1), \rho(\partial_2))$

(2002 $H = T$ (any G), 2009 any H, G ($k=1$), 2011 in general)

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Thm. $R = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$\cong G^{2g}$, $\mu = [A_1, B_1] \cdots [A_g, B_g] : R \to G$

$[a,b] = aba^{-1}b^{-1}$

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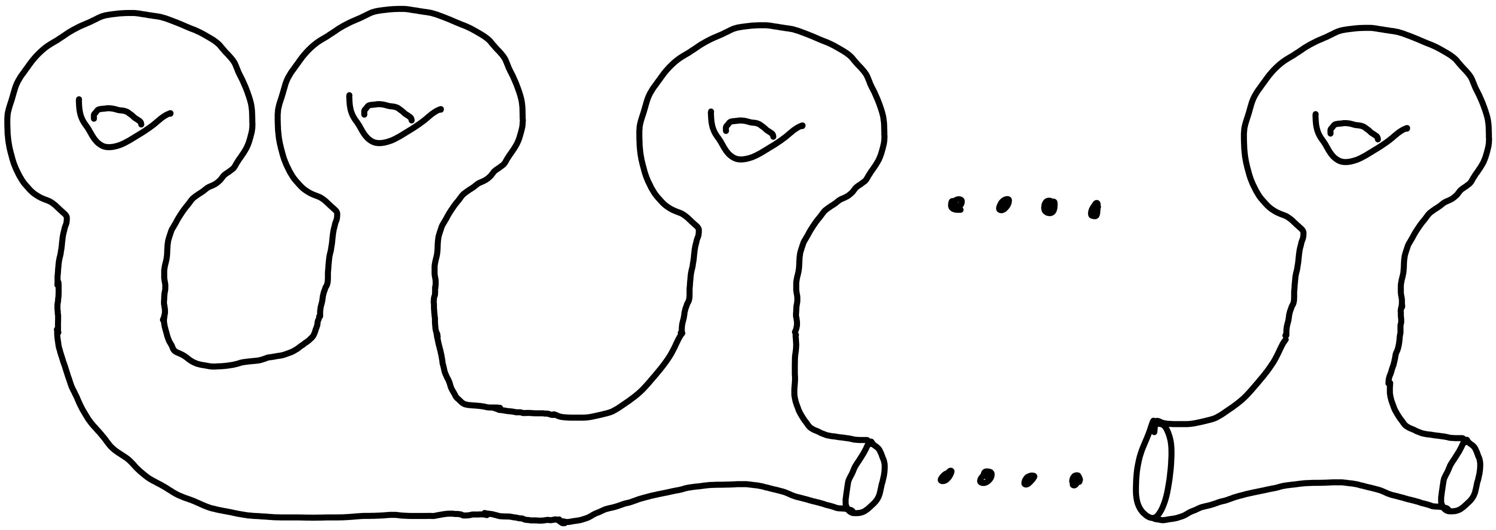

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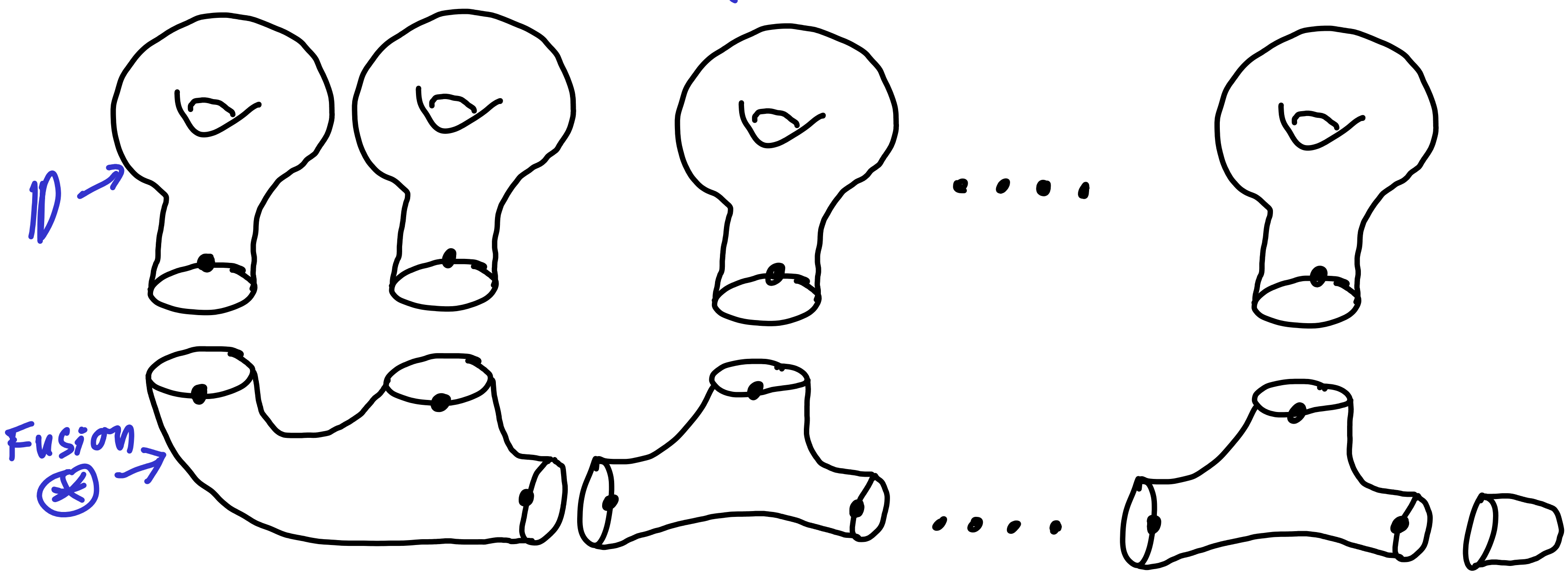
Cor.
- $\mathcal{M}_B = R/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$

E.g. $\mathcal{M}_B(\Sigma_g) = R/\!/G = \mu^{-1}(1)/G = \{A,B \in G^{2g} \mid \prod [A_i,B_i] = 1\}/G$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



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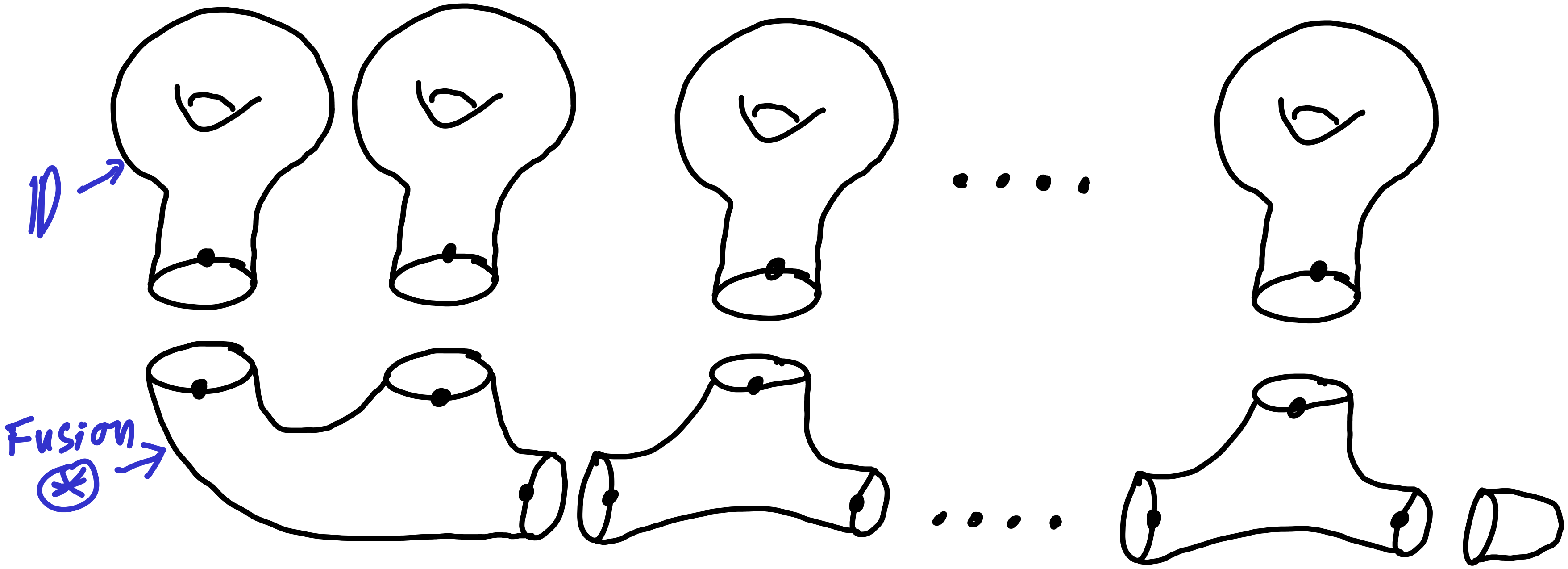
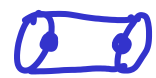
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$
- Can fuse simple pieces: $R = \mathbb{D} \circledast \cdots \circledast \mathbb{D}$, $\mathbb{D} = R(\Sigma_{1,1})$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)

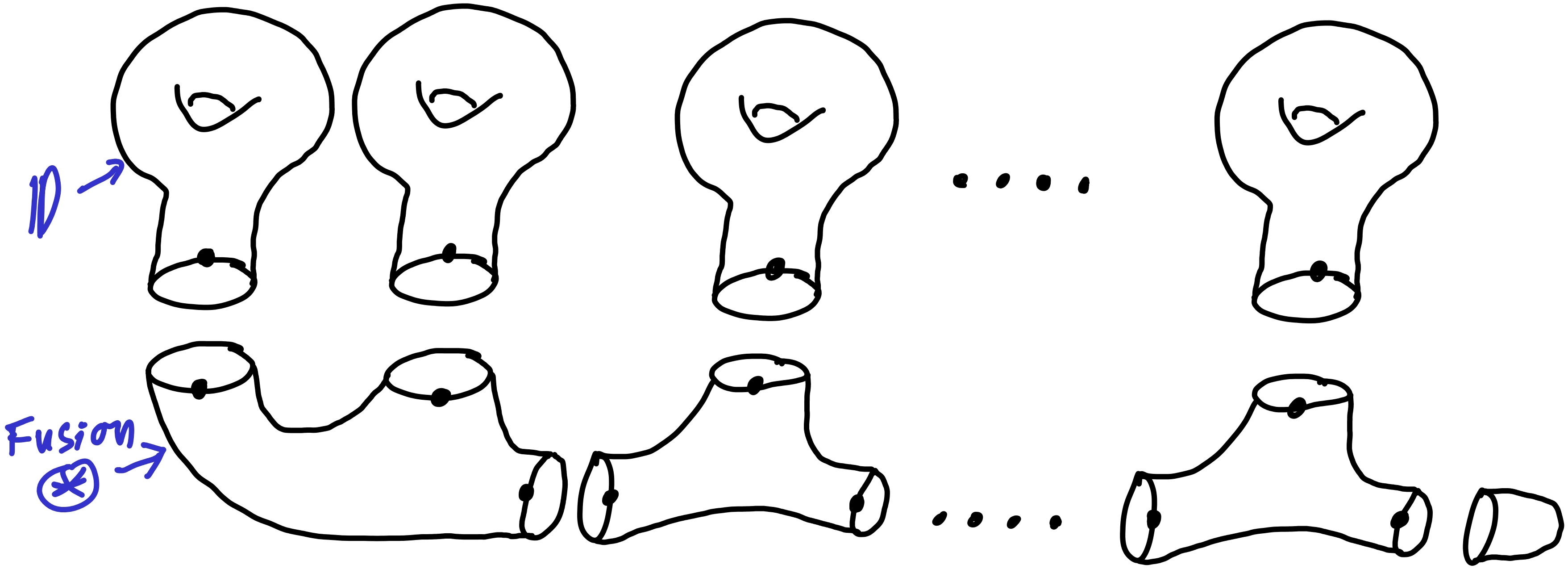


Toolbox:

- $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G \times G$, • $\mathcal{C} \subset G$
- $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}($  $) \cong G \times G$ "double"
- $\circledast$ fusion , • $\oslash$ reduction ($/\!/G$)

$$\mathcal{M}_B(\underset{\sim}{\mathcal{C}}) = \mathbb{D} \circledast \cdots \circledast \mathbb{D} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/ G$$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Toolbox: • $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G\times G$, • $\mathcal{C} \subset G$

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• $\circledast$ fusion , • $\oslash$ reduction ($/\!/G$)

Now add fission spaces $\mathcal{A} = {}_G\mathcal{A}^k_H$ $\forall\, G, H, k$

$\Rightarrow$ lots of new algebraic symplectic/Poisson varieties

"fission varieties" $\supsetneq$ (untwisted) wild character varieties

Wild character varieties

E.g. Birkhoff 1913 wrote presentations in generic setting:

$$(C_1^{-1} h_1 S_{2k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{2k_m}^{(m)} \cdots S_1^{(m)} C_m) = 1$$

(see Jimbo–Miwa–Ueno 1981 equation 2.46)

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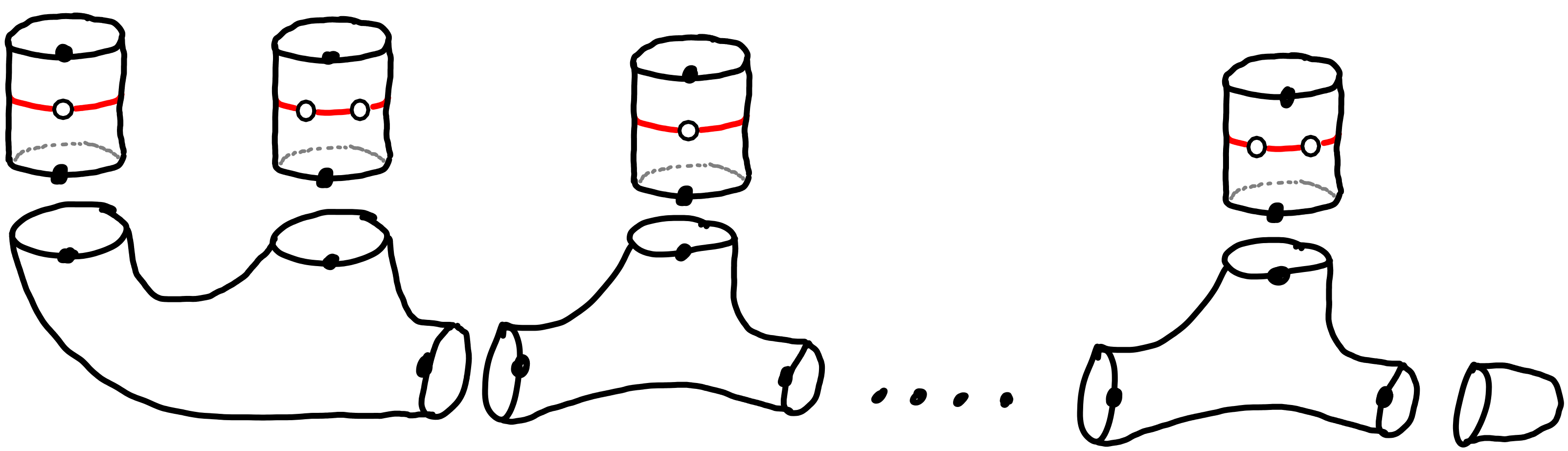
$$\mathcal{R} = {}_G\mathcal{A}_T^{k_1} \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_2} \underset{G}{\circledast} \cdots \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_m} \xrightarrow{\mu} T^m \times G$$



Thm Reductions with fixed $h_i \in T$ are symplectic

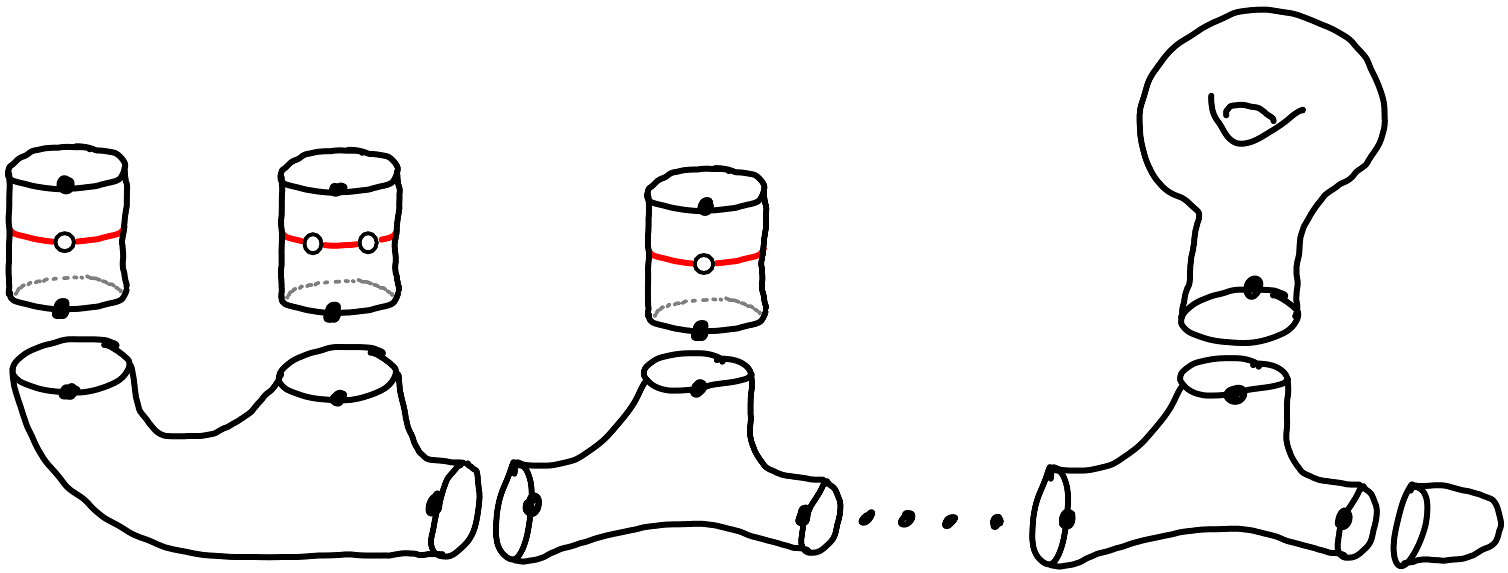
(Adv. Math. 2001 "irreg. Atiyah Bott", algebraic quasi-Hamiltonian approach 2002)

Wild character varieties

Similarly in general ($\sim$ any alg. connections on twisted G-bundles)

$$(C_1^{-1}h_1 S^{(1)}_{k_1}\cdots S^{(1)}_1 C_1)\cdots(C_m^{-1}h_m S^{(m)}_{k_m}\cdots S^{(m)}_1 C_m)\overrightarrow{\prod_1^g} A_iB_iA_i^{-1}B_i^{-1}=1$$

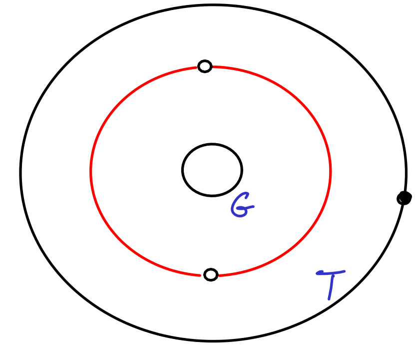
$$\mathrm{THom}_{\mathbb{S}}(\Pi,G)=\mathcal{A}_1\circledast\cdots\circledast\mathcal{A}_m\circledast \mathbb{D}^{\circledast g}/\!/G$$



<u>Thm</u> Wild character variety $\mathcal{M}_B=\mathrm{THom}_{\mathbb{S}}(\Pi,G)/\underset{\sim}{H}$ is a Poisson variety with symplectic leaves got by fixing (twisted) conjugacy classes of formal monodromy

... An. Inst Fourier '09, arXiv:1111.6228, <u>arXiv:1512.08091 (with D. Yamakawa)</u>

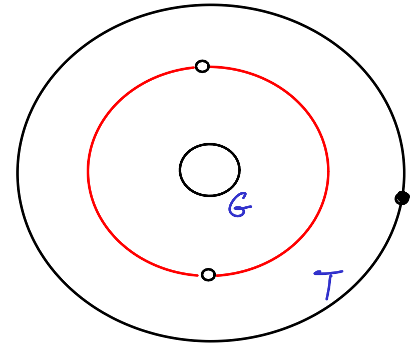
Wild character varieties



E.g. $\quad {}_G\mathcal{A}^1_T/G \cong T \times U_+ \times U_-$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Wild character varieties



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Thm (Drinfeld / Semenov Tian Shansky, DeConcini Procesi 1993)

$U_q(\mathfrak{g})$ quantizes a Poisson variety $G^* \cong T \times U_+ \times U_-$

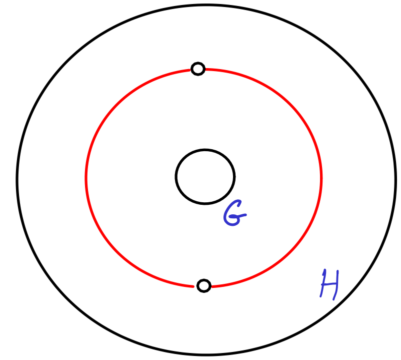
Thm (PB Invent. Math 2001)

$G^* \cong {}_G\mathcal{A}^1_T/G$ as a Poisson variety

Cor. The Drinfeld-Jimbo quantum group is modular

(comes from moduli of connections on curves)

Wild character varieties



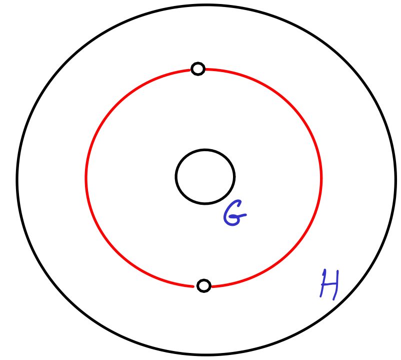
E.g. $\quad {}_G\mathcal{A}^1_H / G\times H \cong (H \times U_+ \times U_-)/H$

is an algebraic Poisson variety with symplectic leaves

$$\mathcal{M}_B(\mathcal{C}, \check{\mathcal{C}}) = \{ h, S_1, S_2 \mid h \in \check{\mathcal{C}},\ h S_1 S_2 \in \mathcal{C} \}/H$$

for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Wild character varieties



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Thm (Fourier-Laplace, Malgrange 1991)

This class of varieties $\cong$ all tame genus zero character varieties

Thm – symplectic structures match too (PB arXiv 1307)

– and the hyperkähler metrics (Sz. Szabo arXiv 1407)

$\rightsquigarrow$ notion of "representations" of abstract moduli space

Plato to Painlevé

(McKay – Harnad) c.f. PB 0706.2634

Sakai's question

Exercise 3

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groups: Tetra. Octa. Icosa. $\subset$ $SO_3(\mathbb{R})$

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$$\uparrow$$

binary groups: $\tilde{T}$ $\tilde{O}$ $\tilde{I}$ $\subset$ $SU_2 \subset SL_2(\mathbb{C})$

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$$\uparrow \qquad \uparrow \qquad \uparrow$$

resolve: X_T X_O X_I

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$$\uparrow \qquad \uparrow \qquad \uparrow$$

resolve: $\tilde{X}_T$ $\tilde{X}_O$ $\tilde{X}_I$

+deform

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$\mathbb{C}^6$ $\mathbb{C}^7$ $\mathbb{C}^8$

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$\circlearrowleft$ $\circlearrowleft$ $\circlearrowleft$

Weyl groups: $W(E_6)$ $W(E_7)$ $W(E_8)$

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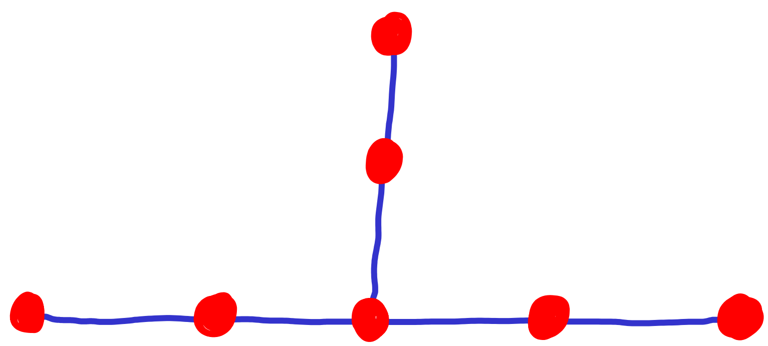
singularities: $\mathbb{C}^2/\tilde{T}$ $\mathbb{C}^2/\tilde{O}$ $\mathbb{C}^2/\tilde{I}$

resolve + deform: $\mathbb{C}^2/\tilde{T} \leftarrow X_T \rightsquigarrow \mathbb{C}^6$, $\mathbb{C}^2/\tilde{O} \leftarrow X_O \rightsquigarrow \mathbb{C}^7$, $\mathbb{C}^2/\tilde{I} \leftarrow X_I \rightsquigarrow \mathbb{C}^8$

Weyl groups: $\circlearrowleft W(E_6)$ $\circlearrowleft W(E_7)$ $\circlearrowleft W(E_8)$

Kronheimer (1989):
- smooth fibres are complete hyperkähler 4-folds
- construct in terms of affine Dynkin graph

E.g. E_6 case (hol. symplectic approach)



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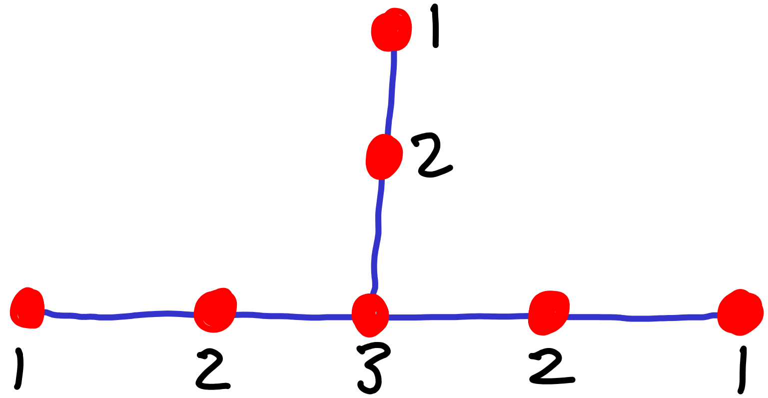
Γ



$\underset{\sim}{d}$

$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

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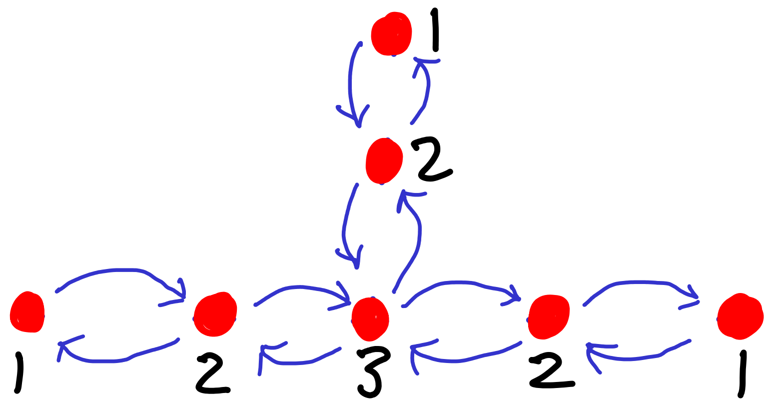

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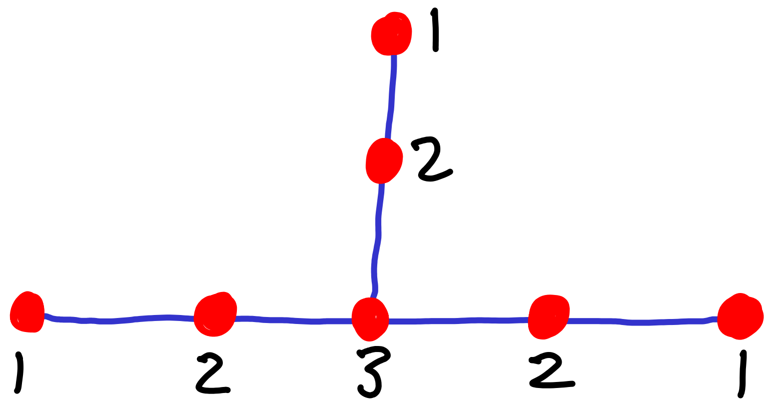

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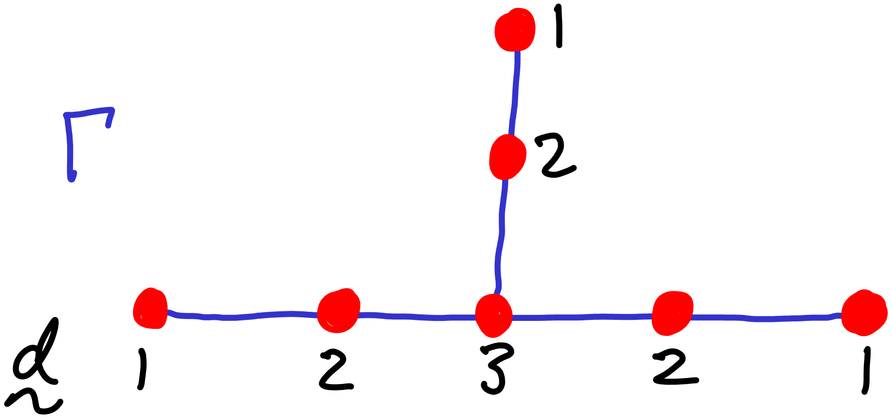

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$$G = GL(\mathbb{C}^{\underset{\sim}{d}}) = \prod GL_{d_i}(\mathbb{C})$$

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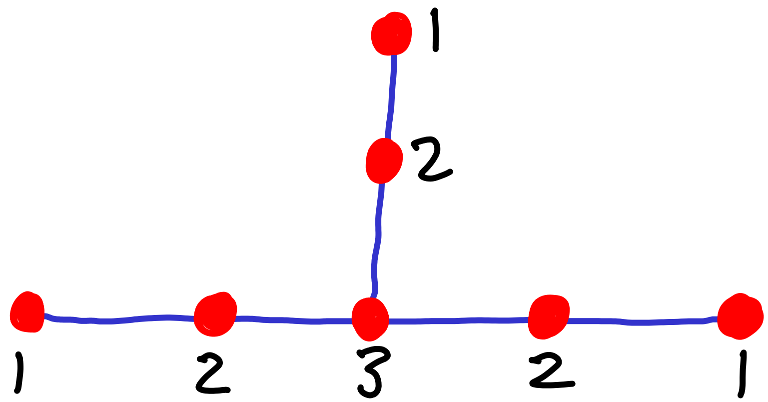

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$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

$$\mathcal{G} = GL(\mathbb{C}^{\underset{\sim}{d}}) = \prod GL_{d_i}(\mathbb{C})$$

$$\mathcal{N} = NQV(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) = \mathbb{V} /\!\!/_{\underset{\sim}{\lambda}} \mathcal{G} = \mu^{-1}(\underset{\sim}{\lambda}) / \mathcal{G}$$

$$\underset{\sim}{\lambda} \in Lie(\mathcal{G})^* \cong \prod End(\mathbb{C}^{d_i}) \quad \text{central}$$

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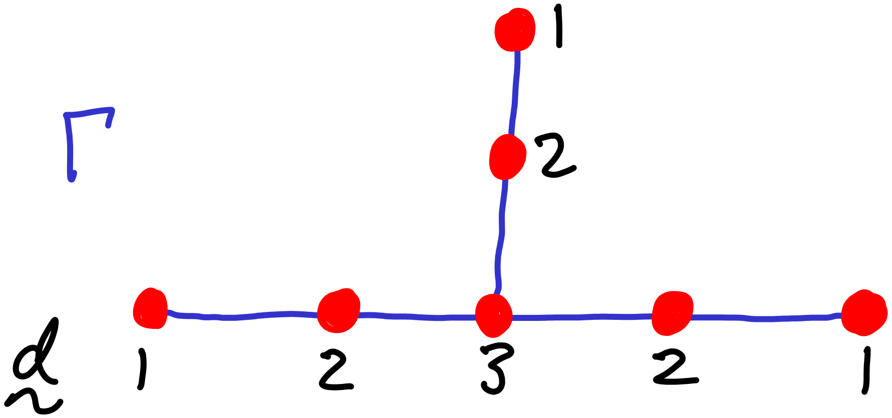
Γ



$$\boxed{\dim_{\mathbb{C}}(\mathcal{N}) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})}$$

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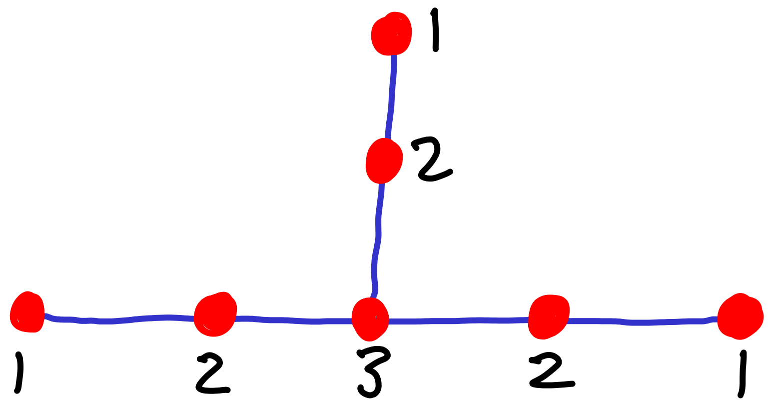

$\underset{\sim}{d}$

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$\mathcal{N}$ is modular $\cong$ moduli space of Fuchsian systems $\mathcal{M}^*$

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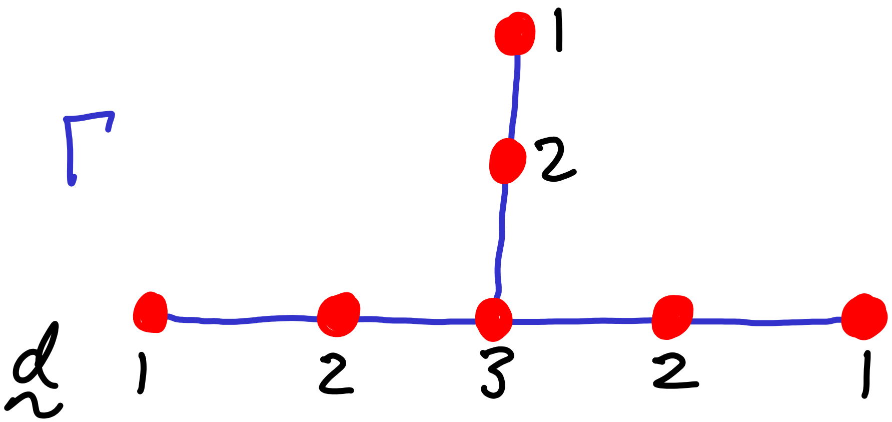

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$$\cong \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 /\!\!/ GL_3(\mathbb{C}) \qquad \begin{pmatrix} \mathcal{O}_i \subset \mathfrak{gl}_3(\mathbb{C}) \\ \text{coadjoint orbit} \\ \dim 6 \end{pmatrix}$$

$$\dim_{\mathbb{C}} = 6 + 6 + 6 - 2(9-1) = 2$$

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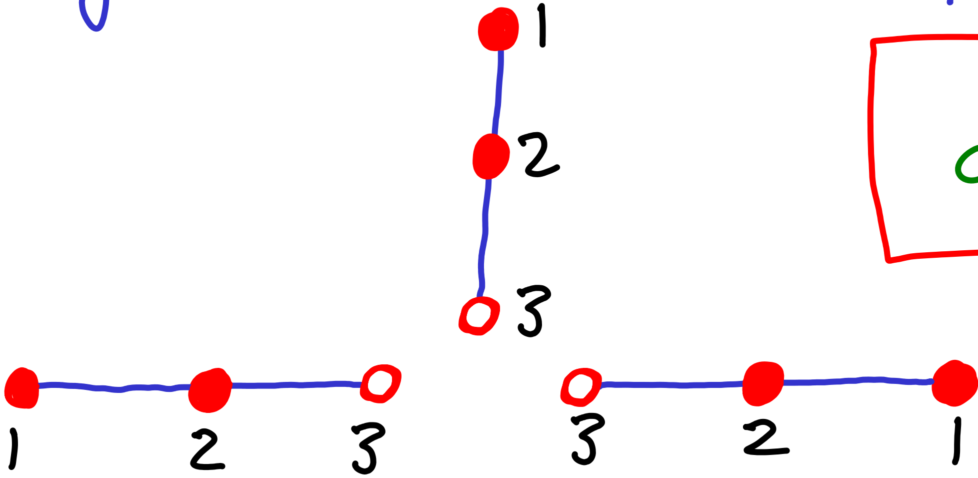

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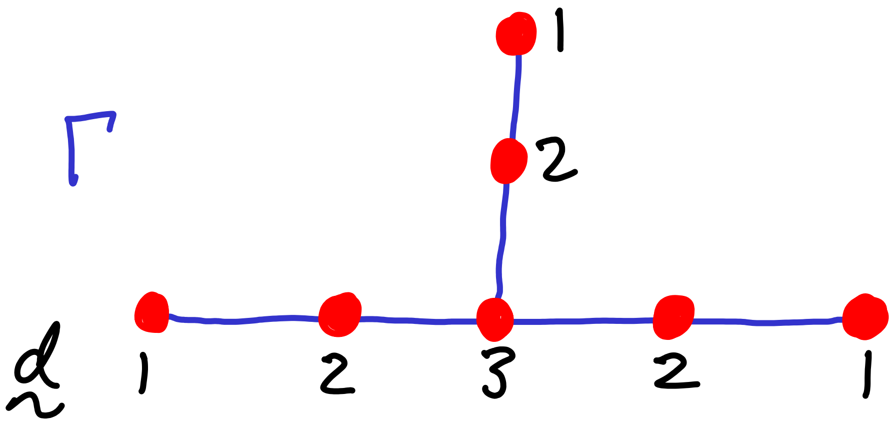

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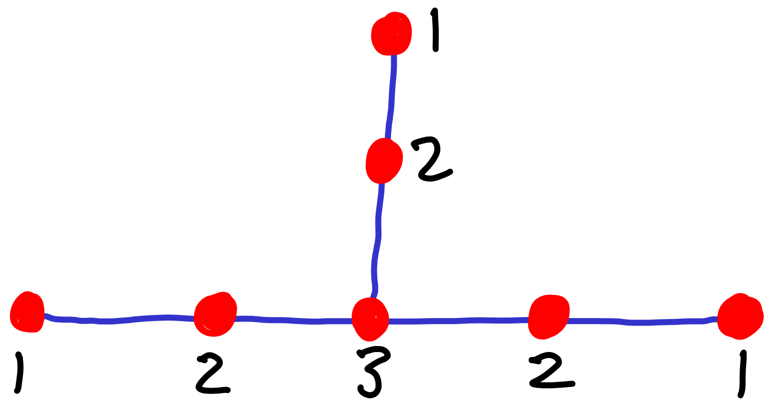

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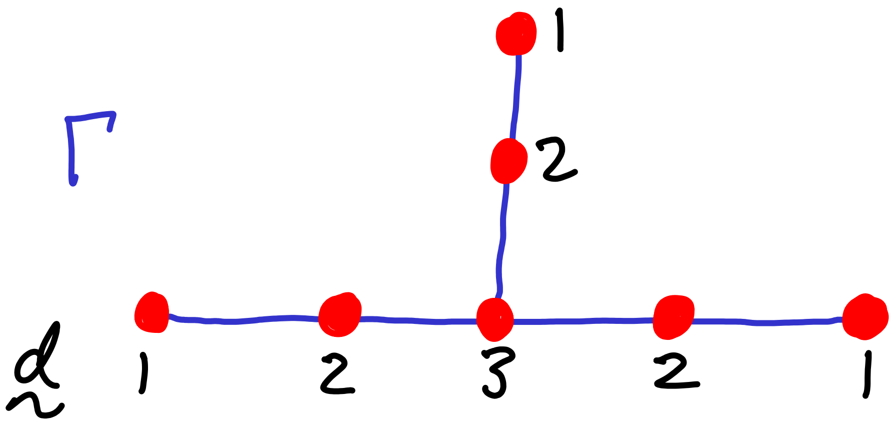
$$\nabla = d - \left(\frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \frac{A_3}{z-a_3} \right) dz \quad , \quad A_i \in \mathcal{O}_i$$

E.g. E_6 case (hol. symplectic approach)



$$\boxed{\dim_{\mathbb{C}}(N) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})}$$

N is <u>modular</u> $\cong$ moduli space of Fuchsian systems $\mathcal{M}^*$

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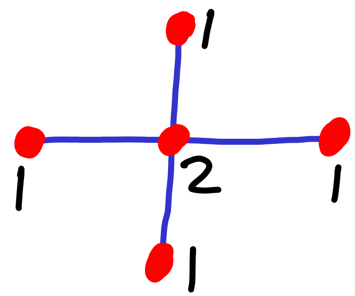
- NQV of any star-shaped Γ is modular (Kraft-Procesi, Nakajima, Crawley-Boevey)
- Get multiplicative version = character variety $\mathcal{M}_B \cong \mathcal{C}_1 \circledast \mathcal{C}_2 \circledast \mathcal{C}_3 /\!/ GL_3$

$$\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow{RH} \mathcal{M}_B \qquad \text{"Global Lie theory"}$$

$\exists$ one more star-shaped affine Dynkin graph:

$\hat{D}_4$



$\sim$ quaternion group $\subset SU_2$

$\{\pm 1, \pm i, \pm j, \pm k\}$

$W(D_4) \circlearrowleft \mathbb{C}^4$ "constants"

Rank 2 Fuchsian systems with 4 poles $\rightsquigarrow$ cross ratio $\in \mathcal{M}_{0,4}$

"modular parameters" / "times"

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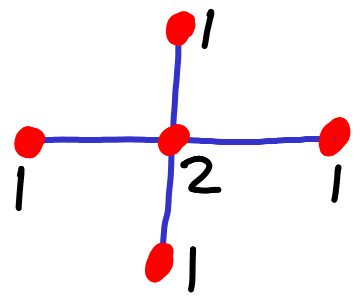

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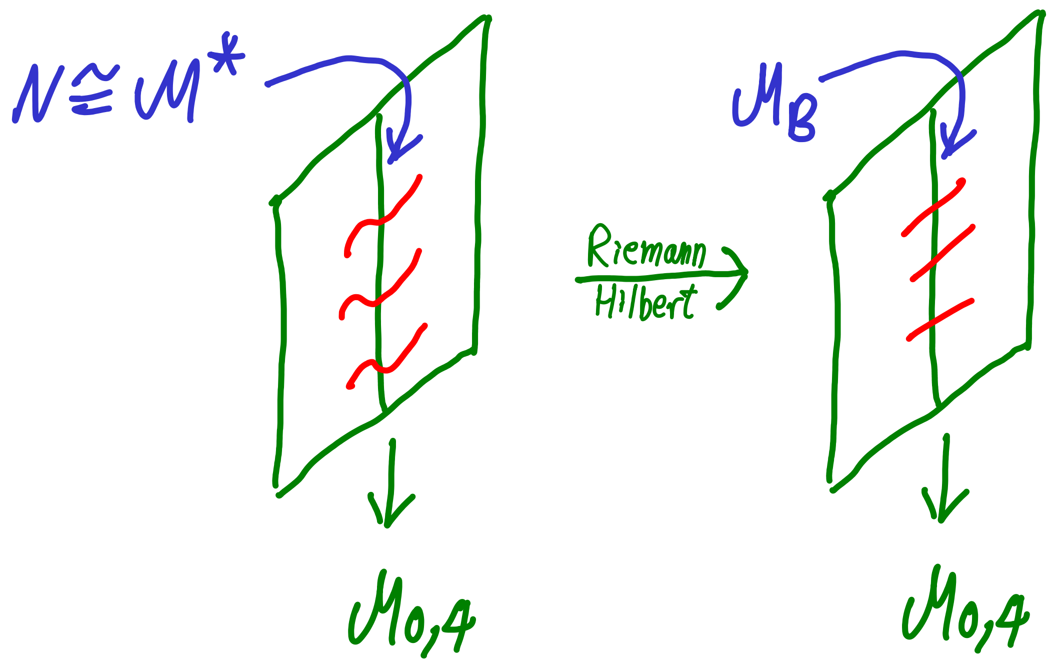

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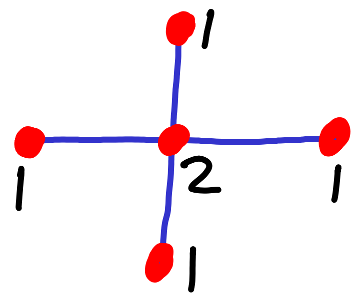

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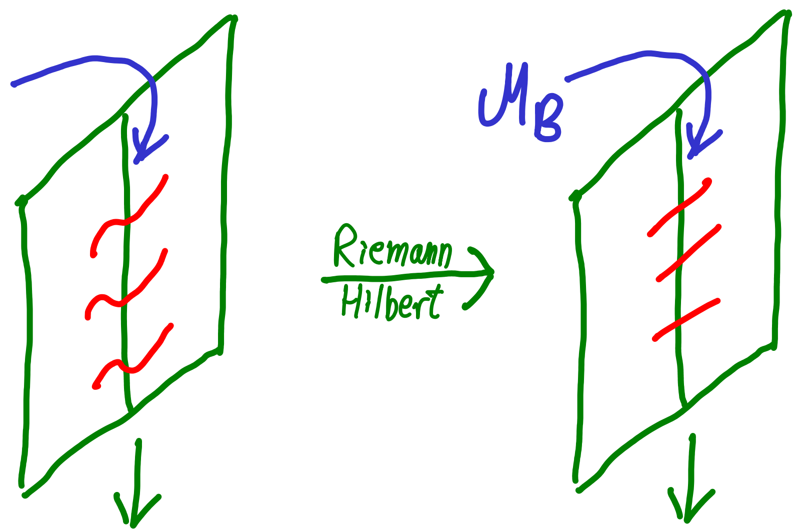
$N \cong \mathcal{M}^*$ $\qquad$ $\mathcal{M}_B$



$\mathcal{M}_{0,4}$ $\qquad$ $\mathcal{M}_{0,4}$

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $t \in \mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\}$

$\mathcal{M}_B \cong$ Fricke-Klein-Vogt cubic surface

$$xyz + x^2 + y^2 + z^2 = ax + by + cz + d$$

$\exists$ one more star-shaped affine Dynkin graph:

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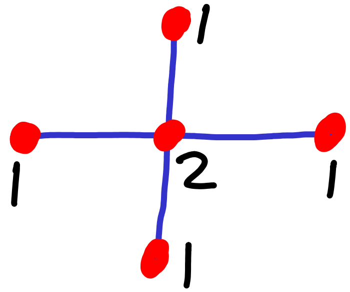

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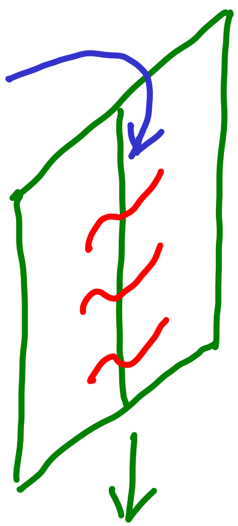
Okamoto 1987: affine Weyl group $W(\hat{D}_4) \circlearrowright \mathbb{C}^4$ relating $P\underline{VI}$ equations

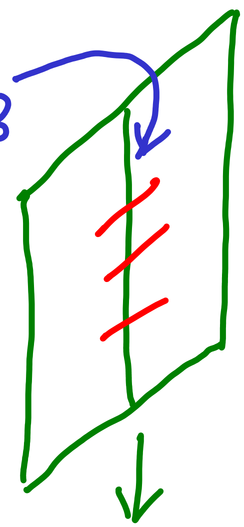
$N \cong \mathcal{M}^*$



$\mathcal{M}_{0,4}$

$\xrightarrow[\text{Hilbert}]{\text{Riemann}}$

$\mathcal{M}_B$



$\mathcal{M}_{0,4}$

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$$- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y'$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

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THE PAINLEVÉ EQUATIONS AND THE DYNKIN DIAGRAMS

Kazuo Okamoto

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Tokyo, Japan

1 Painlevé Systems

Let δ be a differential on $\mathbf{C}(t)$, i.e.

$$\delta = f(t)\frac{d}{dt},$$

$f(t)$ being a rational function in t, and

$$\mathrm{H}(t; q, p) \in \mathbf{C}[t, q, p],$$

a polynomial in three variables (t, q, p). We consider the Hamiltonian system of ordinary differential equations:

$$\begin{aligned} \delta q &= \frac{\partial \mathrm{H}}{\partial p}, \\ \delta p &= -\frac{\partial \mathrm{H}}{\partial q}, \end{aligned} \tag{1}$$

under the assumption that H is of the second degree with respect to p. Therefore, by

P_J	1	2	3	4	5	6
δ	$\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$t(t-1)\frac{d}{dt}$
number of parameters	0	1	2	2	3	4
Affine Weyl Group	--	$\mathbf{A}_1$	$\mathbf{B}_2$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{D}_4$
Particular solutions	--	Airy	Bessel	Hermite-Weber	Confluent Hyper-geometric	Gauß' Hyper-geometric

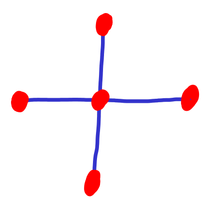
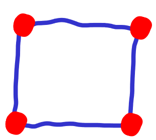
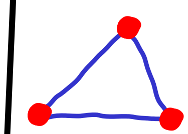
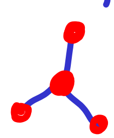
(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

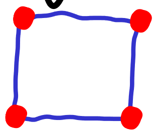
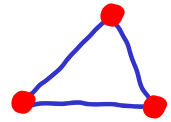
$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

(ALE space of type $\hat{A}_3, \hat{A}_2, \hat{A}_1$)

(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) & (\text{ALE space of type } \hat{A}_3, \hat{A}_2, \hat{A}_1) \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, $rk\,2$)	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram				?		?
Special solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$?	Weber ?	Bessel-Clifford ${}_0F_1$?	Airy ?	–

Questions ① What are the higher dimensional modular quiver varieties lying over    generalising the stars?

② What about Painlevé 1 & Painlevé 3 ($\mathcal{M}^*(P_3) \ncong NQV(\Gamma) \ \forall \Gamma$) & their higher dimensional analogues?

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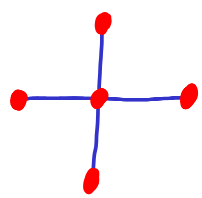
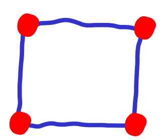
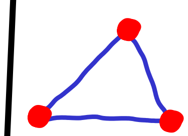
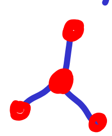
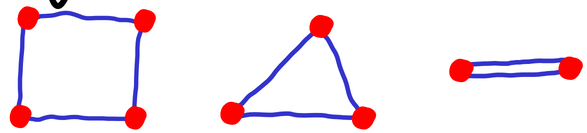
Diagram				?		?
Special Solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$?	Weber ?	Bessel-Clifford ${}_0F_1$?	Airy ?	–

Questions ① What are the higher dimensional modular quiver varieties lying over  generalising the stars?

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& their higher dimensional analogues?

---

③ What is the 'deeper' analogue of $\mathcal{M}_{0,4}$ in general?

⇝ moduli of wild Riemann surfaces

④ What is the 'deeper' analogue of the nonlinear local system $\mathcal{M}_B \to \mathcal{M}_{0,4}$?

⇝ local system of wild character varieties over any admissible deformation of a wild Riemann surface

[P.B. Annals of Math. 2014]

Very good connections

$\sim$ models in Biquard–B. 2004

(cf. exposition in $\begin{cases} \text{arXiv:1203.6607} \\ \text{arXiv:1703.10376} \end{cases}$)

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall\, a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(*\underset{\sim}{a})$ meromorphic connection

such that ...

Very good connections

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such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $\nabla = d - A$, $\quad A = dQ + \Lambda \frac{dz}{z}$ + holomorphic terms

  $Q = \sum_1^k \frac{A_i}{z^i}$, A_i diagonal matrices (irregular type)

  $\Lambda \in \mathfrak{h}$ preserves $\mathcal{F}_a$, $\mathfrak{h} = \mathrm{Lie}(H)$, $H = C_G(Q)$

["Good" if some local cyclic pullback is very good (twisted case)]

$\leadsto \mathcal{M}_{DR}$ moduli of stable connections, $\underset{\sim}{Q}$, $Gr(\Lambda)$, parabolic weights fixed

Very good Higgs bundles

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Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall\, a \in \underset{\sim}{a}$)

$\Phi : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic Higgs field

such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $$\Phi = dQ + \Lambda \frac{dz}{z} + \text{holomorphic terms}$$

$$Q = \sum_1^k \frac{A_i}{z^i}, \quad A_i \text{ diagonal matrices} \quad (\text{irregular type})$$

$$\Lambda \in \mathfrak{h} \text{ preserves } \mathcal{F}_a, \quad \mathfrak{h} = \mathrm{Lie}(H),\ H = C_G(Q)$$

["Good" if some local cyclic pullback is very good (twisted case)]

$\rightsquigarrow \mathcal{M}_{\mathrm{Dol}}$ moduli of stable Higgs bundles, $\underset{\sim}{Q}$, $\mathrm{Gr}(\Lambda)$, parabolic weights fixed

Choices: • $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ "wild Riemann surface" (modular parameters)

• eigenvalues/orbit of Λ, parabolic weights (constants)

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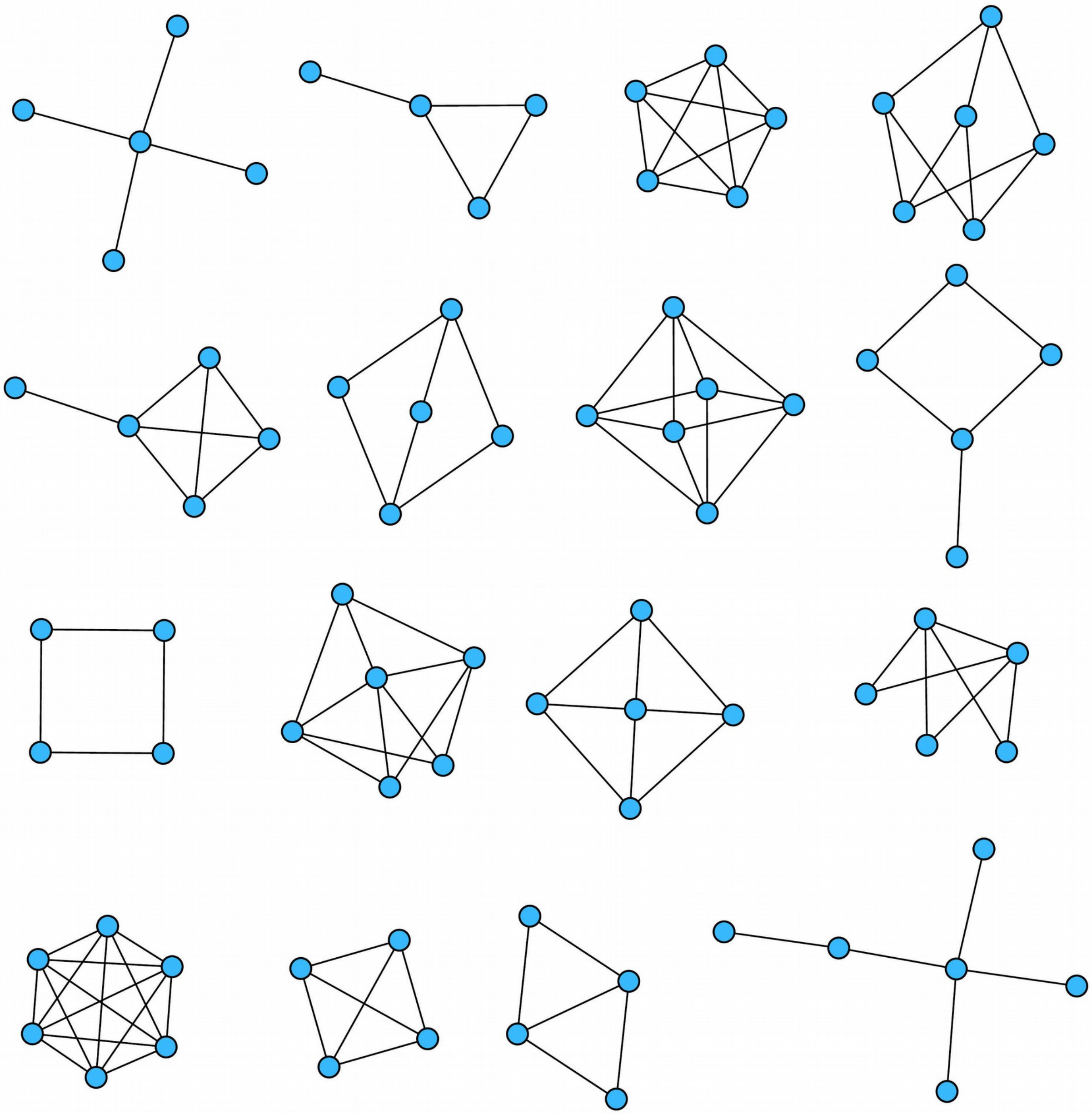
"supernova graphs" (core + legs)

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in x\mathbb{C}[x]$$

core nodes $= \{q_i\}$, $\#\text{edges}(q_i, q_j) = \deg(q_i - q_j) - 1$
+ legs from $\Lambda \in \mathfrak{h} = \prod \mathfrak{gl}_{d_i}(\mathbb{C})$



Idea $\mathcal{M}^* \cong \mathcal{O} /\!/ G$

$\begin{cases} dQ + \Lambda \frac{dz}{z} \in \mathcal{O} \subset \mathfrak{g}_k^* \\ G_k = GL_n(\mathbb{C}[z]/z^{k+1}) \end{cases}$

$\cong H \backslash\!\backslash_{\Lambda} \widetilde{\mathcal{O}} /\!/ G$

"extended orbit" $\widetilde{\mathcal{O}} \circlearrowleft G \times H$

$1 \to B_k \to G_k \xrightarrow{ev} G \to 1$

$\cong H \backslash\!\backslash_{\Lambda} \mathcal{O}_B$

$\mathcal{O}_B \subset \mathfrak{b}_k^*$ Birkhoff orbit

B_k-coadjoint orbit of dQ

decoupling: $\boxed{\widetilde{\mathcal{O}} \cong (T^*G) \times \mathcal{O}_B}$

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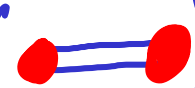
E.g. $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$, $\mathcal{O}_B \cong T^*\mathbb{C}^2 = \mathbb{W}($  $)$ (Painlevé II)

$\mathcal{M}^* \cong$ Eguchi-Hanson space ($\hat{A}_1$ ALE space) $T^*\mathbb{P}^1$, $\mathcal{O} \subset sl_2(\mathbb{C})$

Qn 2

Thm (B–Yamakawa 2020)

$\exists$ uniform way to define a diagram for any meromorphic connection on $\mathbb{P}^1$ with ≤ 1 irreg. singularity

- $\dim(\mathcal{M}_B) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})$ form from Cartan matrix C of diagram
- Can have loops/edges of negative multiplicity

[Any moduli space on $\mathbb{P}^1$ has such a representation]

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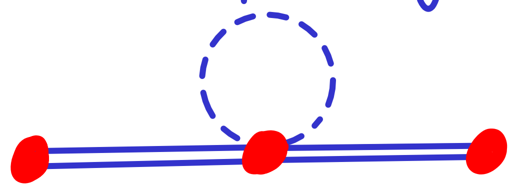
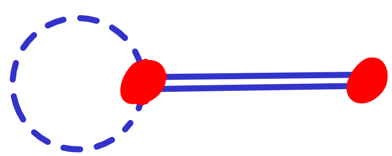
e.g. Painlevé III



$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\underset{\sim}{d} = (1, 1, 1)$$

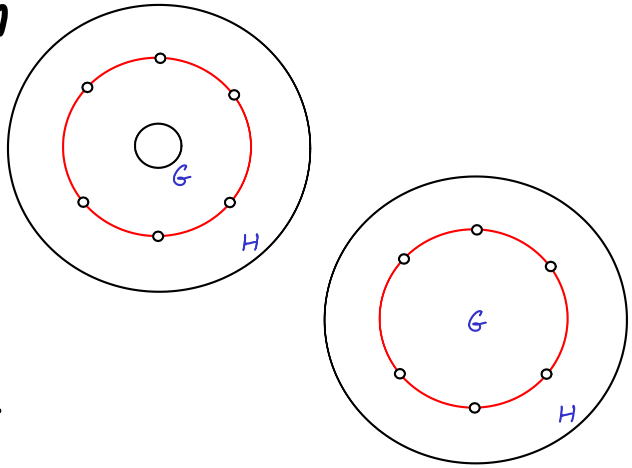
- $\cong_{\mathbb{Z}}$ Intersection form of $\mathcal{M}_{DR}$
- Weyl group $\cong W_{aff}(B_2)$ as Okamoto had
- special solutions (Bessel-Clifford)



Qn 2 Idea: pass to wild character variety / & use general presentations of them

$$\mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \overset{RHB}{\cong} \mathcal{M}_B$$ wild character variety

	Hamiltonian	quasi-Hamiltonian
$G\times H$-spaces:	$\tilde{\mathcal{O}}$	$\mathcal{A}$
H-spaces:	$\mathcal{O}_B$	$\mathcal{B} = \mathcal{A}/\!\!/G$
G-spaces:	$\mathcal{O}$	$\mathcal{C} = \mathcal{A}/\!\!/_{\gamma} H$



"deeper conjugacy classes" ($\gamma = e^{2\pi i \Lambda}$)

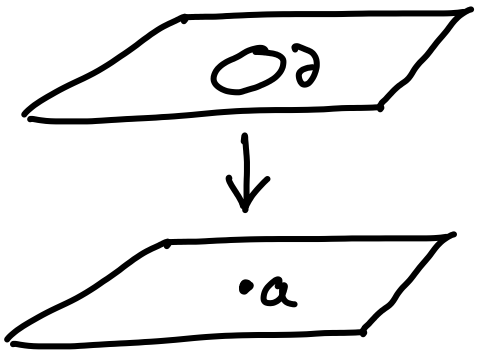
can do all this side in general twisted case (B-Y 2015)
+ looks like quiver rep. for GL_n

General choices / boundary data (twisted case) [Betti weights zero]

Fact $\exists$ covering $\mathcal{I} \to \partial$ such that:

$$\{\text{connections on formal punctured disk}\} \Longleftrightarrow \{\mathcal{I}\text{-graded local systems of vector spaces}\}$$

[Fabry, Hukuhara, Turrittin, Levelt, Deligne]



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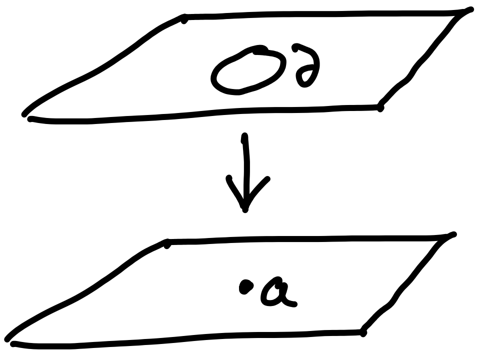

function on sector: $q = \sum_{i>0} a_i z^{-i/r}$ $(r \in \mathbb{N})$

$\Longrightarrow$ circle $\langle q \rangle$ (Riemann surface / Galois orbit)

$$\begin{array}{c}\langle q \rangle\\ \downarrow\\ \partial\end{array}$$

$$\boxed{\begin{array}{c}\tilde{I} = \bigcup \langle q \rangle\\ \downarrow\\ \partial\end{array} \quad \text{exponential local system}}$$

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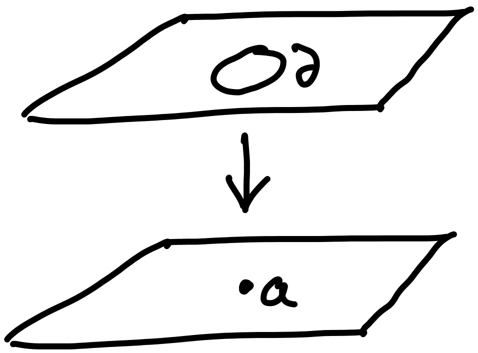

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$\mathcal{I}$-graded local system $V \to \partial$ of vector spaces

$\iff$ local system $V \to \mathcal{I}$ with compact support

i.e. $V \to I$, $I \subset \mathcal{I}$ finite subcover

$\Rightarrow$ Irregular class $\Theta = n_1\langle q_1 \rangle + \dots + n_m \langle q_m \rangle$ $\quad n_i = \mathrm{rk}\, V|_{\langle q_i \rangle}$

$+$ monodromy classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$

- points of maximal decay $\partial \subset \mathcal{I}$

$\partial(q) \subset \langle q \rangle$ where e^q max decay

- Irregularity: $\mathrm{Irr}(q) = \# \partial(q)$

$\mathrm{Irr}(\sum n_i \langle q_i \rangle) = \sum n_i \mathrm{Irr}(q_i)$

- Ramification: $\mathrm{Ram}(q) = \deg \pi : \langle q \rangle \longrightarrow \partial$ (min r)

Choose $\Theta = \sum n_i \langle q_i \rangle$, $e_i \subset GL_{n_i}(\mathbb{C})$, at $\infty \in \mathbb{P}^1$

wild Riemann surface $(\mathbb{P}^1, \infty, \Theta)$

- points of maximal decay $\partial \subset \Sigma$

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Core diagram: nodes $\sim \{\langle q_i \rangle\}$

$\#$ arrows $\langle q_i \rangle \to \langle q_j \rangle = B_{ij} := \begin{cases} A_{ij} - \beta_i \beta_j & i \neq j \\ A_{ii} - \beta_i^2 + 1 & i = j \end{cases}$

$A_{ij} := \mathrm{Irr}(\mathrm{Hom}(\langle q_i \rangle, \langle q_j \rangle))$, $\beta_i = \mathrm{Ram}(q_i)$

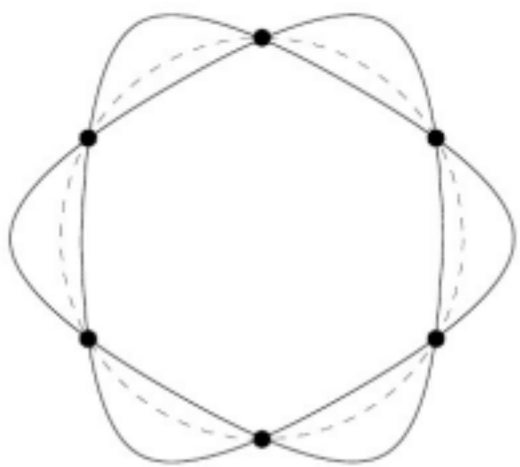
(symmetrized) Cartan matrix: $\boxed{C = 2 - B}$

Then glue on legs from classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$ as before

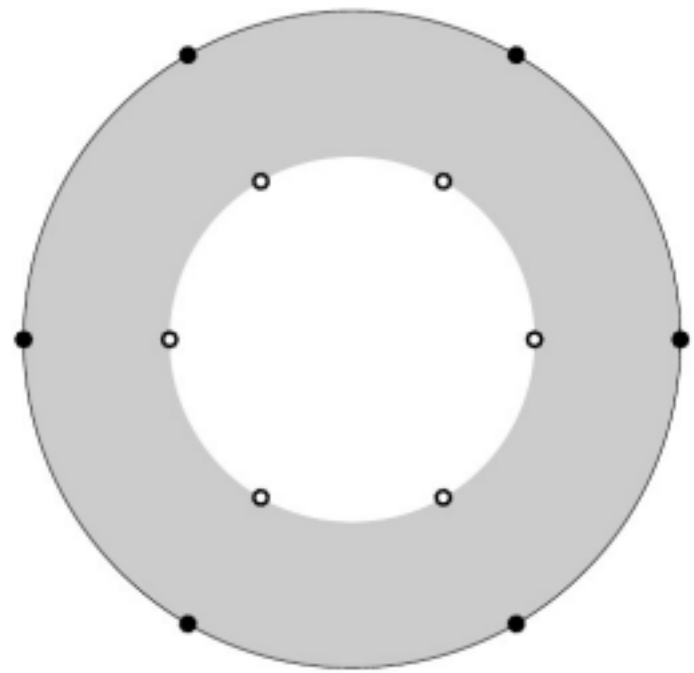
Painlevé II: $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$

solutions involve e^Q

plot growth/decay of $\exp(x^3)$, $\exp(-x^3)$:



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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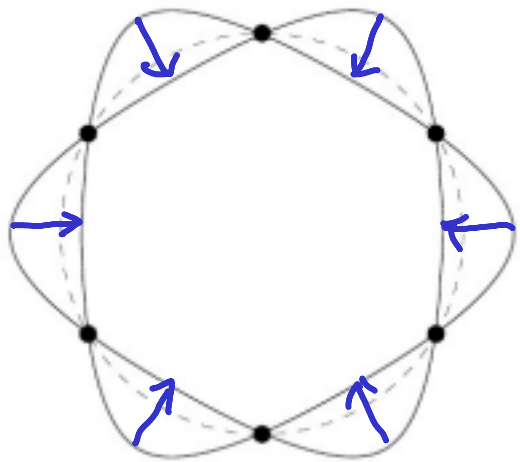

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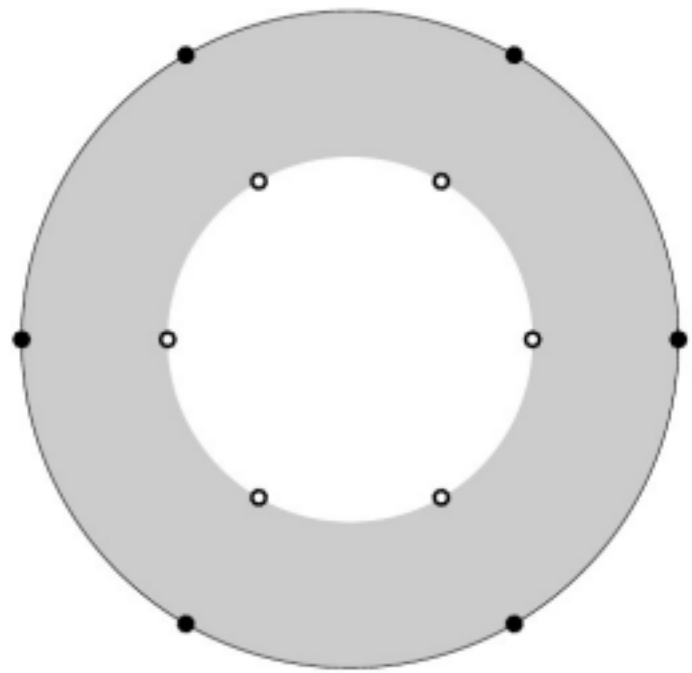

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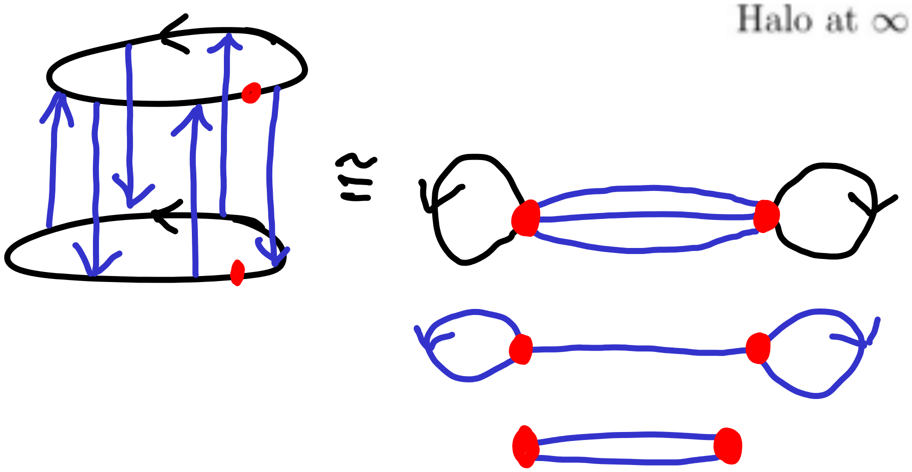


$\mu_G = 1$

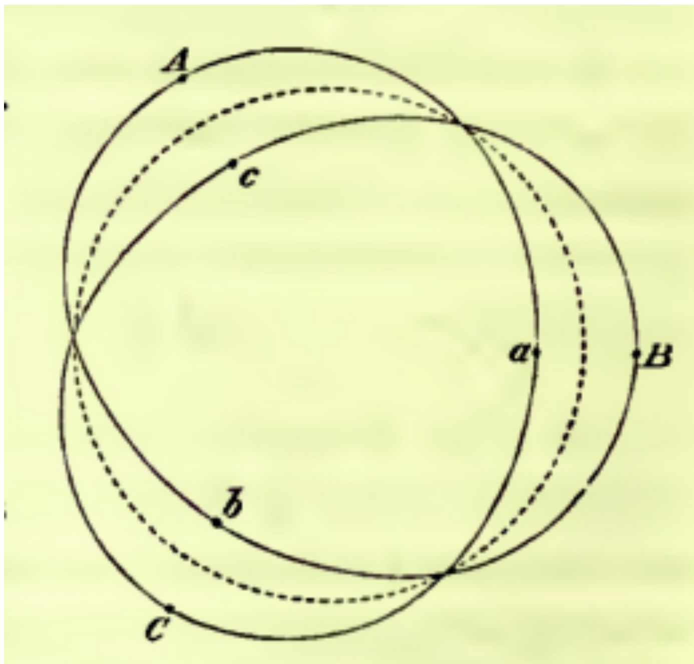
$h S_6 S_5 \dots S_1 = 1$

2x2 matrix relation

result: $\hat{A}_1$

Airy equation (Stokes 1857)

solutions involve $\exp(x^{3/2})$



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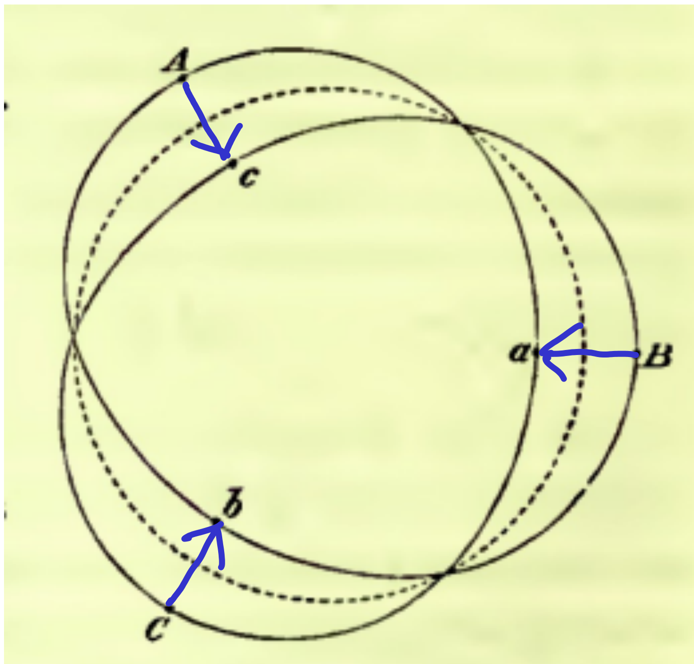
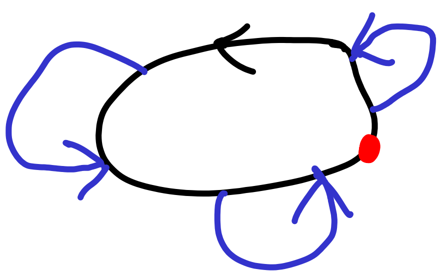
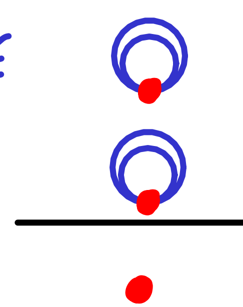


$$\mu_G = 1$$

$$\underbrace{h}_{\begin{pmatrix}0 & *\\ 1 & 0\end{pmatrix}}\ \underbrace{S_3}_{u_+}\ \underbrace{S_2}_{u_-}\ \underbrace{S_1}_{u_+} = 1$$



$\cong$

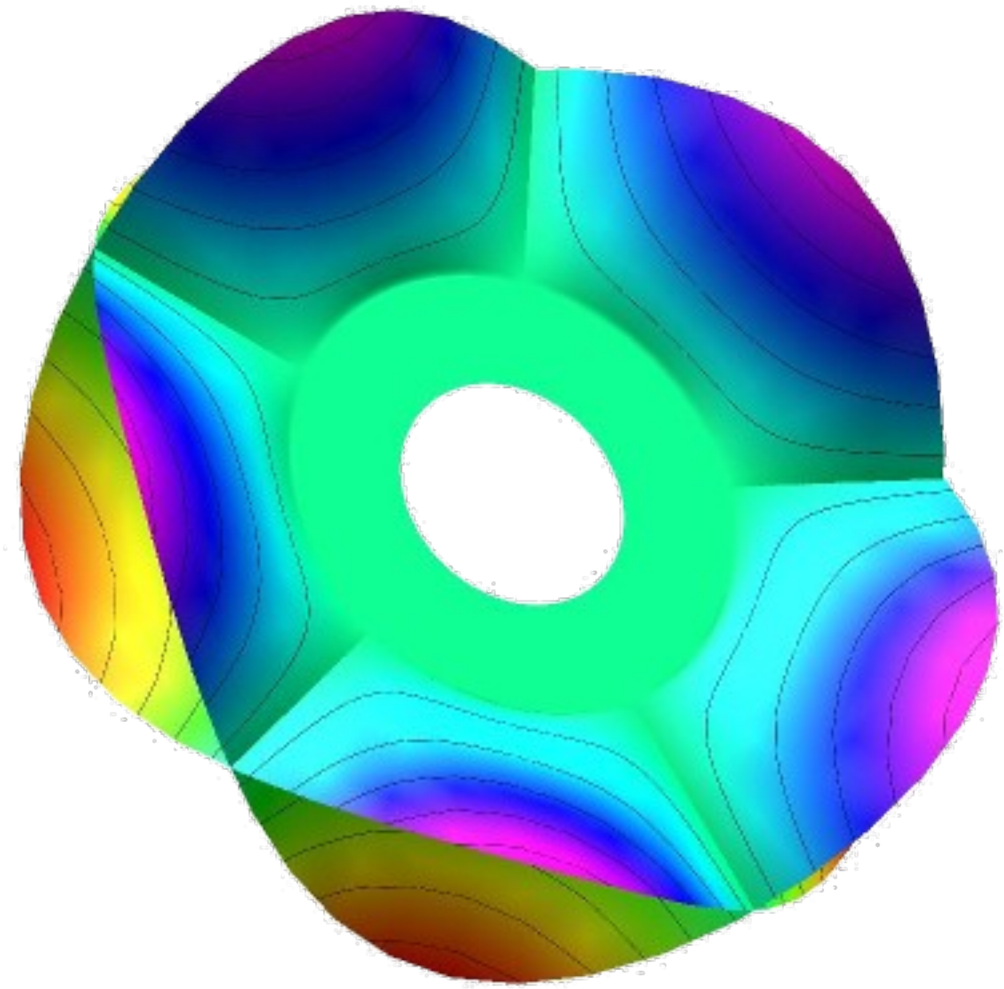


relations

resulting diagram

Painlevé 1

$\exp(x^{5/2})$



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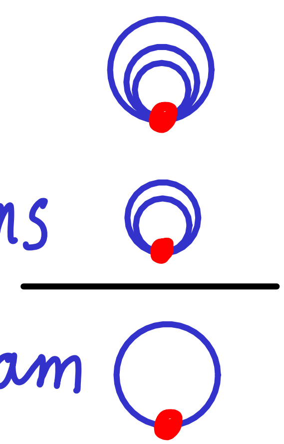


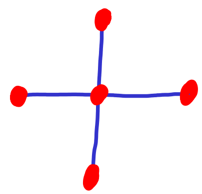
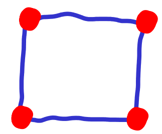
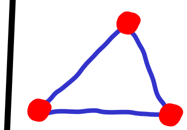
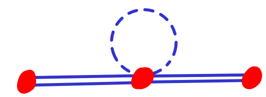
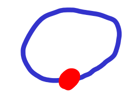
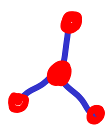
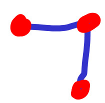
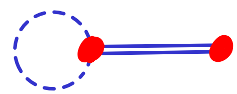
Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, $rk\,2$)	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram						
Special solutions	Gauss $_2F_1$ 	Kummer $_1F_1$ 	Weber 	Bessel-Clifford $_0F_1$ 	Airy 	–

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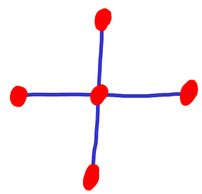
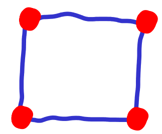
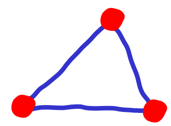
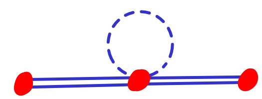
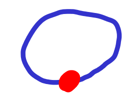
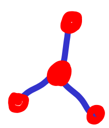
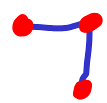
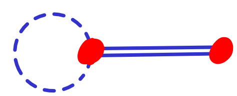
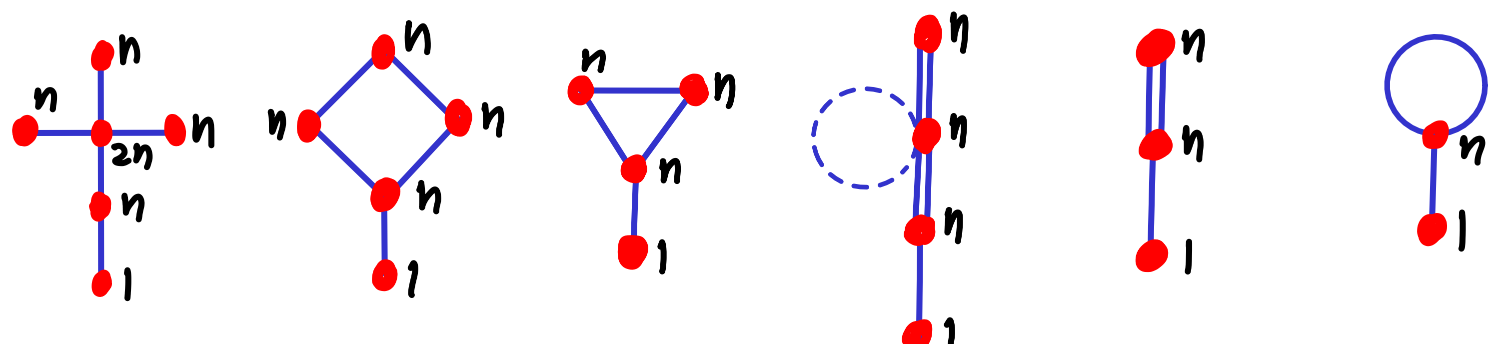
Diagram						
Special solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$ 	Weber 	Bessel-Clifford ${}_0F_1$ 	Airy 	—

Higher Painlevé spaces:



dim $2n$, conjecturally $\cong \mathrm{Hilb}^n(2d\,\mathcal{M}_B)$ (known by Groechenig in tame case)